International Journal of Theoretical Physics, Vol. 35, No. 10, 1996

Poisson Brackets and the Feynman Problem

Paul Bracken[1]

Received April 17, 1996

A derivation of a pair of Maxwell equations which is based on the concept of a Poisson structure on a manifold is given. The idea is geometric in character, and is extended to a generalized algebra. The special case of the dynamics for a particle in a Yang–Mills field is obtained as a consequence of the generalized case.

1. INTRODUCTION

The Poisson bracket has played an essential role in the development of classical mechanics and the transition to quantum mechanics. Suppose that one has a classical dynamical system Γ, and one would like to know if Γ can be given a description in terms of some Poisson bracket and a Hamiltonian function. This has come to be known as the inverse problem of Poisson dynamics. One typically has to make some further assumptions concerning the type of Poisson manifold. If the inverse problem is solved and a Poisson tensor $\mathbf{T}$ is found, one can then ask if there exists a symplectic realization for it, that is, a symplectic manifold (P, Ω) and a Poisson map $\Phi: P \rightarrow M$ (Marsden and Ratiu, 1994; Abraham and Marsden, 1978).

Feynman's procedure to develop minimal coupling to standard electrodynamics is related to these areas and has come to be known as Feynman's problem (Dyson, 1990). Although the original intent was in finding new kinds of particle dynamics, it has evolved into a problem which deals with a full set of dynamical systems, in particular, those which are second-order differential equations on a velocity phase space, **TQ**. The problem can then be stated in a general way as finding all Poisson tensors on **TQ** such that they have Hamiltonian vector fields which correspond to second-order differential equations and such that $\{q^i, q^j\} = 0$, that is, the property of localizability holds (Carinena *et al.*, 1995).

[1]Quantum Theory Project, University of Florida, Gainesville, Florida 32611.

The idea of the paper is to give a presentation of Feynman's original idea. Moreover, it has developed into a derivation of the Lorentz force law and two of Maxwell's equations. It will be illustrated that in order to obtain dynamics, one need only postulate the Poisson brackets of the particle. It will then be shown that this can be generalized to the case of the dynamics of particles which possess other internal degrees of freedom, I^a, but do not necessarily generate a finite algebra. The case in which the I^a generate a Lie algebra will be written down. In a similar manner, one need postulate the particle's Poisson brackets and a Hamiltonian evolution. Not everything here is new; the main intent is to give a mathematically complete presentation of the development, and to present some physical insights into the mathematics. It will be shown that a derivation which is very algebraic as well as geometric in nature can be developed. This type of examination may be useful, given the current interest in gauge theories, quantum algebras, and their use in constructing gauge theories, as well as the current interest in duality (Tanimura, 1992; Kaku, 1988; Hatfield, 1992).

To establish some notation and background, let us call $\mathcal{F}$ the algebra of classical observables on the manifold M. A Poisson structure on a manifold M is a skew-symmetric bilinear map which is denoted $\{\ ,\ \}$: $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ such that:

(i) $(\mathcal{F}, \{\ ,\ \})$ satisfies the Jacobi identity,

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \tag{1}$$

(ii) The map $X_F = \{\cdot, F\}$ is a derivation of the associative algebra $\mathcal{F}(\mathcal{M})$ on M, that is, it satisfies the Leibnitz rule,

$$\{F, GH\} = G\{F, H\} + \{F, G\}H \tag{2}$$

A manifold M which is endowed with a Poisson bracket on $\mathcal{F}(\mathcal{M})$ is called a Poisson manifold.

In the following, the properties (i) and (ii) will be used frequently. A deep theory can be developed around this concept. For example, let P be a Poisson manifold. If $H \in \mathcal{F}(\mathcal{P})$, then there is a unique vector field X_H on P such that

$$X_H G = \{G, H\} \tag{3}$$

for all $G \in \mathcal{F}(\mathcal{P})$. The vector field X_H is called the Hamiltonian vector field of H. This is a consequence of the fact that any derivation on $\mathcal{F}(\mathcal{P})$ is represented by a vector field.

Any function $H \in \mathcal{F}$ will define a dynamical system on M by the equation

$$\frac{dF}{dt} = \{F, H\} \tag{4}$$

Finally, in a set of local coordinates (w^a) for M, the coordinate expression for the Poisson bracket $\{F, G\}$ is

$$\{F, G\} = X_G F = \{w^a, G\} \frac{\partial F}{\partial w^a} \tag{5}$$

Repeated indices are summed over.

2. POISSON BRACKETS AND THE MAXWELL EQUATIONS

A derivation which was originally proposed by Feynman (Dyson, 1990) to generate two of the Maxwell equations is presented in detail. The presentation which is given here tries to rely as much as possible on the basic algebraic and analytic properties of the bracket which have been written down in the introduction. The idea is that no further assumptions in this section are made regarding the bracket other than these and the Leibnitz rule for the time derivative.

Let the local coordinate variables be written in the form $(w^a) = (x^i, v^i)$ for $i = 1, 2, 3$, where the x^i may be interpreted as position coordinates and v^i represent velocity. First of all, the fundamental Poisson brackets are postulated to be as follows:

$$\{x_i, x_j\} = 0, \qquad m\{x_i, v_j\} = \delta_{ij} \tag{6}$$

The equations of motion which are based on these variables using (4) are given as follows:

$$\dot{x}^i = \{x^i, H\} = v^i, \qquad m\dot{v}^i = m\{v^i, H\} = F^i \tag{7}$$

Notice that this implies that the Hamiltonian dynamical system is a second-order differential system.

Differentiating the second bracket in (6) with respect to time gives

$$\{\dot{x}_i, v_j\} + \{x_i, \dot{v}_j\} = 0 \tag{8}$$

Multiplying both sides by m and using the equations of motion, one obtains

$$m\{\dot{x}_i, \dot{x}_j\} + \{x_i, F_j\} = 0 \tag{9}$$

Since the bracket is bilinear, this equation can be put into the form

$$\{\{x_i, F_j\}, x^k\} + m\{\{\dot{x}_i, \dot{x}_j\}, x^k\} = 0 \tag{10}$$

Substituting $\dot{x}^i$, $\dot{x}_j$, and x^k into Jacobi's identity, one obtains

$$\{\{\dot{x}^i, \dot{x}_j\}, x^k\} + \{\{\dot{x}_j, x^k\}, \dot{x}^i\} + \{\{x^k, \dot{x}^i\}, \dot{x}_j\} = 0$$

Since the bracket $\{\dot{x}_j, x^k\}$ is proportional to δ_j^k, this reduces to the constraint

$$\{\{\dot{x}_i, \dot{x}_j\}, x_k\} = 0$$

Substituting this into (10), one obtains

$$\{x^k, \{x^i, F_j\}\} = 0$$

The tensor $\{x_i, F_j\}$ is therefore antisymmetric on account of the bracket property. This can be expressed in its dual form by a relation such as

$$\{F_i, x_j\} = \frac{1}{m} \epsilon_{ijk} H_k \tag{11}$$

where H_k is the component of a pseudotensor $\mathbf{H}$ which will depend on the coordinates of M and possibly time.

It has been shown that $\{x^k, \{x^i, F_j\}\} = 0$, so when the relation above is substituted, a bracket which contains H_k can be obtained,

$$\{x^l, H_k\} = 0$$

On account of (6), this means that the vector $\mathbf{H}$ depends only on the position and time of the particle. The equation above and (6) imply that F^i is at most linear in the velocities, so we may write

$$F_i(x) = E_i(x) + \epsilon_{ijk} v^j H^k(x)$$

This is the Lorentz force law when the electric charge is unity. It defines the electric field, and so, using bilinearity and the derivation property, we obtain

$$\begin{aligned}
\{F_i, x_l\} &= \{E_i + \epsilon_{ijk} v^j H^k, x_l\} \\
&= \{E_i, x_l\} + \{\epsilon_{ijk} v^j H^k, x_l\} \\
&= \{E_i, x_l\} + \epsilon_{ijk} v^j \{H^k, x_l\} + \epsilon_{ijk} \{v^j, x_l\} H^k \\
&= \{E_i, x_l\} + \frac{1}{m} \epsilon_{ilk} H^k
\end{aligned}$$

The vectors $\mathbf{E}$ and $\mathbf{H}$ are not independent. On account of equation (11), the last equation above implies that

$$\{E_i, x_l\} = 0$$

This implies that the $\mathbf{E}$ vector, like the $\mathbf{H}$ vector, depends only on the position coordinate and time. Summarizing the two equations which are needed to proceed with the development, one has

$$\{x_i, F_j\} = -m\{\dot{x}_i, \dot{x}_j\}, \qquad \{F_i, x_j\} = \frac{1}{m} \epsilon_{ij}^k H_k$$

Combining these two equations leads to a new equation for H_k in terms of the bracket:

$$\epsilon_{ij}^{k} H_k = m^2\{\dot{x}_i, \dot{x}_j\} \tag{12}$$

Using the identity $\epsilon_{ijk}\epsilon^{ilk} = 2\delta_j^l$ and moving the ϵ-tensor to the right-hand side produces

$$H_i = \tfrac{1}{2}m^2\epsilon_{ikj}\{\dot{x}_k, \dot{x}_j\}$$

Applying the Jacobi identity to the variables $\dot{x}_l$, $\dot{x}_j$, and $\dot{x}_k$ and then contracting indices with ϵ^{jkl} gives

$$\epsilon^{jkl}\{\dot{x}_l, \{\dot{x}_j, \dot{x}_k\}\} = 0$$

Replacing $\{\dot{x}_j, \dot{x}_k\}$ with H_k yields

$$\{\dot{x}_l, H_l\} = 0$$

Using equation (5) and the fact that H_l does not depend on $\dot{x}_i$, one obtains the first important result,

$$\{H_l, \dot{x}_l\} = \{w_a, \dot{x}_l\}\frac{\partial H_l}{\partial w_a} = \{x_a, \dot{x}_l\}\frac{\partial H_l}{\partial x_a} = m^{-1}\frac{\partial H_l}{\partial x_l}$$

The bracket on the left is zero, and so this gives the first Maxwell equation, $\nabla\cdot\mathbf{H} = 0$.

To obtain the second equation, one begins with the equation

$$H_i = \tfrac{1}{2}m^2\epsilon_{ikj}\{\dot{x}_k, \dot{x}_j\}$$

Differentiating both sides of this equation with respect to t, one obtains

$$\frac{\partial H_i}{\partial t} + \frac{\partial H_i}{\partial x_j}\dot{x}^j = \frac{1}{2}m^2\epsilon_{ikj}\{\ddot{x}_k, \dot{x}_j\} + \frac{1}{2}m^2\epsilon_{ikj}\{\dot{x}_k, \ddot{x}_j\} = m^2\epsilon_{kji}\{\ddot{x}_k, \dot{x}_j\}$$

By substituting the equation $F_k = m\ddot{x}_k = E_k + \epsilon_{kal}\dot{x}_a H_l$ from (7), we can write the right-hand side of the equation as

$$\begin{aligned}
&\epsilon_{kji}m\{E_k + \epsilon_{kal}\dot{x}_a H_l, \dot{x}_j\}\\
&\quad = m\epsilon_{ikj}\{E_k, \dot{x}_j\} + m\epsilon_{ikj}\epsilon_{kal}\{\dot{x}_a H_l, \dot{x}_j\}\\
&\quad = m\epsilon_{ikj}\{E_k, \dot{x}_j\} + m\{\dot{x}_j H_i, \dot{x}^j\} - m\{\dot{x}_i H_j, \dot{x}^j\}\\
&\quad = m\epsilon_{ikj}\{E_k, \dot{x}_j\} + m\{H_i, \dot{x}^j\}\dot{x}_j + m\{\dot{x}_j, \dot{x}^j\}H_i - m\{H_j, \dot{x}^j\}\dot{x}_i - m\{\dot{x}_i, \dot{x}^j\}H_j
\end{aligned}$$

The last term on the right-hand side of this equation is zero by symmetry using the equation for H_i, and we substitute $\{H_k, \dot{x}^k\} = 0$. Using (5), this takes the form

$$\frac{\partial H_i}{\partial t} + \frac{\partial H_i}{\partial x_j}\dot{x}^j = -\epsilon_{ijk}\frac{\partial E_k}{\partial x_j} + \frac{\partial H_i}{\partial x^j}\dot{x}^j \tag{13}$$

This is just another Maxwell equation, namely

$$\frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0$$

Notice that although this produces only two of the Maxwell equations, due to the symmetry here, the remaining pair can be obtained by using an elementary transformation which sends **E** into **H** and **H** into $-\mathbf{E}$.

3. GENERALIZED ALGEBRA AND EQUATIONS

The space is again coordinatized by the position coordinates and velocity as (x^i, v^i), where $i = 1, 2, 3$. Suppose the Poisson brackets involving the coordinates are again postulated to be

$$\{x^i, x^j\} = 0, \qquad m\{x^i, v^j\} = \delta^{ij}$$

Introduce internal degrees of freedom which are denoted $I^a = I^a(t)$, $a = 1, \ldots, N$, and postulate the following Poisson bracket relations for them:

$$\{I^a, I^b\} = C^{ab}(I), \qquad \{x^i, I^a\} = 0$$

More generally, assume that for arbitrary functions A and B of the variables x and I, one can write

$$\{A, B\} = C^{ab}(I)\delta_a A \delta_b B \tag{14}$$

in any local region of M, where $\partial_a \equiv \delta_a$ denotes the derivative with respect to I^a. The only assumptions placed on the functions $C^{ab}(I)$ will be

$$C^{ab}(I) = -C^{ba}(I)$$

and

$$\delta_d C^{bc} C^{ad} + \delta_d C^{ca} C^{bd} + \delta_d C^{ab} C^{cd} = 0$$

which follow from the antisymmetry of the Poisson brackets and the Jacobi identity.

In fact, more generally, a convenient way to specify a bracket in finite dimensions is by giving the coordinate relations $\{z^i, z^j\} = B^{ij}(z)$. The Jacobi identity is then implied by the special cases

$$\{\{z^i, z^j\}, z^k\} + \{\{z^k, z^i\}, z^j\} + \{\{z^j, z^k\}, z^i\} = 0$$

which may be equivalent to the differential equations (Marsden and Ratiu, 1994),

$$B^{li}\frac{\partial B^{jk}}{\partial z^l} + B^{lj}\frac{\partial B^{ki}}{\partial z^l} + B^{lk}\frac{\partial B^{ij}}{\partial z^l} = 0$$

It will be shown that the equations of motion for the particle must be of the form

$$m\dot{v}^i = \mathcal{F}^{ij}(x, t, I)v^j + \mathcal{F}^{i0}(x, t, I) \tag{15}$$

$$\dot{I}^a = -A^{ia}(x, t, I)v^i - A^{0a}(x, t, I) \tag{16}$$

The fields satisfy

$$\mathcal{F}^{\mu\nu}(x, t, I) = -\mathcal{F}^{\nu\mu}(x, t, I)$$

and the potentials $A^{\mu a}(x, t, I)$ ($\mu\nu = 0, \ldots, 3$) satisfy the consistency conditions

$$D^{\lambda}\mathcal{F}^{\mu\nu} + D^{\mu}\mathcal{F}^{\nu\lambda} + D^{\nu}\mathcal{F}^{\lambda\mu} = 0 \tag{17}$$

$$\delta_d\mathcal{F}^{\mu\nu}C^{ad} = D^{\mu}A^{\nu a} - D^{\nu}A^{\mu a} \tag{18}$$

$$\delta_d C^{ab}A^{\mu d} = \delta_d A^{\mu b}C^{ad} - \delta_d A^{\mu a}C^{bd} \tag{19}$$

The derivative D^{λ} in these equations is defined by the equation (Stern and Yakushin, 1993),

$$D^{\mu} = \partial^{\mu} - A^{\mu d}\delta_d$$

Here, ∂^0 and ∂^j denote partial derivatives with respect to the coordinates t and x_j, respectively.

Define the two fields $\mathcal{F}^{ij}$ and A^{ia} according to

$$\mathcal{F}^{ij} = -\mathcal{F}^{ji} = m^2\{v^i, v^j\}, \qquad A^{ia} = m\{v^i, I^a\} \tag{20}$$

and apply the Jacobi identities. It will be seen that most of the equations which are required can be obtained using the Jacobi identity. Consider first the variables x^i, v^j, and v^k:

$$\{\{x^i, v^j\}, v^k\} + \{\{v^j, v^k\}, x^i\} + \{\{v^k, x^i\}, v^j\} = 0$$

that is,

$$\{\mathcal{F}^{jk}, x^i\} = 0$$

Similarly, substituting x^i, v^j, and I^a, one obtains

$$\{\{x^i, v^j\}, I^a\} + \{\{v^j, I^a\}, x^i\} + \{\{I^a, x^i\}, v^j\} = 0$$

that is,

$$\{A^{ja}, x^i\} = 0$$

It is for this reason that $\mathcal{F}^{ij}$ and A^{ia} are functions of x, t, and I only.

To begin, substitute the variables I^a, I^b, and v^i into the Jacobi identity to obtain

$$\{\{I^a, I^b\}, v^i\} + \{\{I^b, v^i\}, I^a\} + \{\{v^i, I^a\}, I^b\} = 0$$
$$m\{C^{ab}(I), v^i\} - C^{pa}(I)\delta_p A^{ib} + C^{pb}(I)\delta_p A^{ia} = 0$$
$$-m\{I^p, v^i\}\delta_p C^{ab} + C^{ap}(I)\delta_p A^{ib} - C^{bp}(I)\delta_p A^{ia} = 0$$
$$\delta_p C^{ab} A^{ip} = C^{ap}(I)\delta_p A^{ib} - C^{bp}(I)\delta_p A^{ia}$$

Hence, this is just (19) with $\mu = i$.

Now substitute v^i, v^j, and I^a into the Jacobi identity to obtain

$$\{\{v^i, v^j\}, I^a\} + \{\{v^j, I^a\}, v^i\} + \{\{I^a, v^i\}, v^j\} = 0$$
$$\{\mathcal{F}^{ij}, I^a\} + m\{A^{ja}, v^i\} - m\{A^{ia}, v^j\} = 0$$
$$C^{pa}\delta_p \mathcal{F}^{ij} = m\{A^{ia}, v^j\} - m\{A^{ja}, v^i\}$$

Suppose one substitutes v^i, v^j, and v^k, then the equations which result are

$$\{\{v^i, v^j\}, v^k\} + \{\{v^j, v^k\}, v^i\} + \{\{v^k, v^i\}, v^j\} = 0$$
$$\{\mathcal{F}^{ij}, v^k\} + \{\mathcal{F}^{jk}, v^i\} + \{\mathcal{F}^{ki}, v^j\} = 0$$
$$\{v^k, \mathcal{F}^{ij}\} + \{v^i, \mathcal{F}^{jk}\} + \{v^j, \mathcal{F}^{ki}\} = 0$$

Let us summarize the two equations which were obtained above:

$$\delta_d \mathcal{F}^{ij} C^{ad} = m\{v^j, A^{ia}\} - m\{v^i, A^{ja}\} \tag{21}$$
$$\{v^i, \mathcal{F}^{jk}\} + \{v^j, \mathcal{F}^{ki}\} + \{v^k, \mathcal{F}^{ij}\} = 0$$

Suppose both sides of the equation $m\{x^i, v^j\} = \delta^{ij}$ are differentiated with respect to time using the Leibnitz rule. One obtains an additional equation,

$$\{\dot{x}^i, v^j\} + \{x^i, \dot{v}^j\} = 0$$
$$m^2\{x^i, \dot{v}^j\} = -m^2\{\dot{x}^i, v^j\} = -\mathcal{F}^{ij}$$

Similarly, the bracket $\{x^i, I^a\} = 0$ gives rise to

$$\{\dot{x}^i, I^a\} + \{x^i, \dot{I}^a\} = 0$$
$$m\{x^i, \dot{I}^a\} = -A^{ia}$$

But these imply the structure of equations (15) and (16), which then define A^{0a} and $\mathcal{F}^{i0}$. Taking the time derivative of $\{I^a, I^b\} = C^{ab}(I)$ gives

$$\{\dot{I}^a, I^b\} + \{I^a, \dot{I}^b\} = \delta_d C^{ab}(I)\dot{I}^d = -\delta_d C^{ab}(I)A^{id}v^i - \delta_d C^{ab}(I)A^{0d}$$

Substituting for $\dot{I}^a$ gives

$$\{-A^{ia}v^i - A^{0a}, I^b\} + \{I^a, -A^{ib}v^i - A^{0b}\} = -\delta_d C^{ab}(I)A^{id}v^i - \delta_d C^{ab}(I)A^{0d}$$

that is,

$$-\{A^{ia}v^i, I^b\} - \{A^{0a}, I^b\} - \{I^a, A^{ib}v^i\} - \{I^a, A^{0b}\}$$
$$= -\delta_d C^{ab}(I)A^{id}v^i - \delta_d C^{ab}(I)A^{0d}$$

This can be simplified by using the derivation property, equation (14), and the equation for A^{ia}. It can be seen that one gets (19) with $\mu = 0$, and $\mu = i$, which multiplies v^i.

By differentiating $A^{ia} = m\{v^i, I^a\}$ with respect to time and using the derivation property of the bracket, one obtains

$$\partial^0 A^{ia} + \partial^j A^{ia} v_j + \delta_d A^{ia} \dot{I}^d = m\{\dot{v}, I^a\} + m\{v^i, \dot{I}^a\}$$

Substituting the equations of motion into both sides of this equation, one obtains

$$\partial^0 A^{ia} + \partial^j A^{ia} v_j + \delta_d A^{ia}(-A^{kd}v^k - A^{0d}) = \{\mathcal{F}^{ij}v^j + \mathcal{F}^{i0}, I^a\} + m\{v^i, -A^{ja}v^j - A^{0a}\}$$

Expanding this out, one finds

$$\partial^0 A^{ia} + \partial^j A^{ia} v_j - \delta_d A^{ia} A^{kd} v^k - \delta_d A^{ia} A^{0d}$$
$$= \{\mathcal{F}^{ij}v^j, I^a\} + \{\mathcal{F}^{i0}, I^a\} - m\{v^i, A^{ja}v^j\} - m\{v^i, A^{0a}\}$$
$$= \frac{1}{m}\mathcal{F}^{ij}A^{ja} + \{\mathcal{F}^{ij}, I^a\}v^j + \{\mathcal{F}^{i0}, I^a\} - \frac{1}{m}\mathcal{F}^{ij}A^{ja} - m\{v^i, A^{ja}\}v^j$$
$$- m\{v^i, A^{0a}\}$$
$$= \{\mathcal{F}^{ij}, I^a\}v^j + \{\mathcal{F}^{i0}, I^a\} - m\{v^i, A^{ja}\}v^j - m\{v^i, A^{0a}\}$$
$$= C^{pa}\delta_p \mathcal{F}^{ij}v^j + C^{pa}\delta_p \mathcal{F}^{i0} - m\{v^i, A^{ja}\}v^j - m\{v^i, A^{0a}\}$$

Using the equation for $C^{pa}\delta_p \mathcal{F}^{ij}$, we transform the right-hand side into

$$C^{pa}\delta_p \mathcal{F}^{i0} - m\{v^i, A^{0a}\} - m\{v^j, A^{ia}\}v^j$$

Now equate terms which are linear in v_j and terms which are independent of v_j:

$$\partial^i A^{ja} - \delta_d A^{ja} A^{id} = -m\{v^i, A^{ja}\}$$

or, in terms of the operator D^i,

$$D^i A^{ja} = -m\{v^i, A^{ja}\}$$

The other constraint is

$$\partial^0 A^{ia} - \delta_d A^{ia} A^{0d} = C^{da}\delta_d \mathcal{F}^{i0} - m\{v^i, A^{0a}\}$$

By substituting $D^iA^{ja} = -m\{v^i, A^{ja}\}$, this can be written

$$C^{ad}\delta_d\mathcal{F}^{i0} = A^{0d}\delta_d A^{ia} - \partial^0 A^{ia} - m\{v^i, A^{0a}\} = D^iA^{0a} - D^0A^{ia}$$

But this is just (18) with μ, $\nu = i0$. The fact that $\{v^i, A^{ja}\}$ and $\{v^i, A^{0a}\}$ are independent of v has been used. This follows from the Jacobi identity; for example, using the Jacobi identity and taking x, v, and A^{ia}, one obtains

$$\{x^i, \{v^j, A^{ka}\}\} + \{v^j, \{A^{ka}, x^i\}\} + \{A^{ka}, \{x^i, v^j\}\} = 0$$

Since $\{x^i, v^j\}$ is proportional to δ^{ij} and $\{A^{ka}, x^i\} = 0$, it follows that $\{x^i, \{v^j, A^{ka}\}\} = 0$ and therefore $\{v^j, A^{ka}\} = 0$. Substituting $D^iA^{ja} = -m\{v^i, A^{ja}\}$ into the constraint for $C^{ad}\delta_d\mathcal{F}^{ij}$, one obtains

$$C^{ad}\delta_d\mathcal{F}^{ij} = m\{v^j, A^{ia}\} - m\{v^i, A^{ja}\} = D^iA^{ja} - D^jA^{ia}$$

which is equation (18) with $(\mu\nu) = (ij)$.

By differentiating $\mathcal{F}^{ij} = m^2\{v^i, v^j\}$ with respect to t, one obtains

$$\partial^0\mathcal{F}^{ij} + \partial^k\mathcal{F}^{ij}v_k + \delta_d\mathcal{F}^{ij}\dot{I}^d = m^2\{\dot{v}^i, v^j\} + m^2\{v^i, \dot{v}^j\}$$

Substituting the equations of motion, we find

$$\begin{aligned}
&\partial^0\mathcal{F}^{ij} + \partial^k\mathcal{F}^{ij}v_k + \delta_d\mathcal{F}^{ij}(-A^{kd}v_k - A^{0d})\\
&\quad = m\{\mathcal{F}^{ik}v^k + \mathcal{F}^{i0}, v^j\} + m\{v^i, \mathcal{F}^{jk}v_k + \mathcal{F}^{j0}\}\\
&\quad = m\{\mathcal{F}^{ik}v^k, v^j\} + m\{\mathcal{F}^{i0}, v^j\} + m\{v^i, \mathcal{F}^{jk}v_k\} + m\{v^i, \mathcal{F}^{j0}\}\\
&\quad = m\{v^k, v^j\}\mathcal{F}^{ik} + m\{\mathcal{F}^{ik}, v^j\}v^k + m\{v^i, \mathcal{F}^{jk}\}v_k\\
&\qquad + m\mathcal{F}^{jk}\{v^i, v_k\} + m\{v^i, \mathcal{F}^{j0}\} + m\{\mathcal{F}^{i0}, v^j\}
\end{aligned}$$

Since the bracket is antisymmetric, this equation simplifies to

$$\begin{aligned}
&\partial^0\mathcal{F}^{ij} + \partial^k\mathcal{F}^{ij}v_k - A^{kd}\delta_d\mathcal{F}^{ij}v_k - A^{0d}\delta_d\mathcal{F}^{ij}\\
&\quad = m\{\mathcal{F}^{ik}, v^j\}v^k - m\{\mathcal{F}^{jk}, v^i\}v_k + m\{\mathcal{F}^{i0}, v^j\} - m\{\mathcal{F}^{j0}, v^i\}
\end{aligned}$$

Terms which are linear in v_k and terms which are independent of v_k are to be equated next. This will give two constraint equations as follows; the first is

$$\partial^k\mathcal{F}^{ij} - A^{kd}\delta_d\mathcal{F}^{ij} = m\{\mathcal{F}^{ik}, v^j\} - m\{\mathcal{F}^{jk}, v^i\}$$

This can then be written in terms of the operator $D^\mu = \partial^\mu - A^{\mu d}\delta_d$ as follows:

$$D^k\mathcal{F}^{ij} = m\{v^i, \mathcal{F}^{jk}\} - m\{v^j, \mathcal{F}^{ik}\}$$

$$D^0\mathcal{F}^{ij} = m\{v^i, \mathcal{F}^{j0}\} - m\{v^j, \mathcal{F}^{i0}\}$$

If we take in general $D^if = -m\{v^i, f\}$, we can write the second equation,

$$D^0\mathcal{F}^{ij} = -D^i\mathcal{F}^{j0} + D^j\mathcal{F}^{i0} = -D^i\mathcal{F}^{j0} - D^j\mathcal{F}^{0i}$$

By writing cyclic permutations of the first equation above, we obtain the following three equations:

$$D^k\mathcal{F}^{ij} = m\{v^i, \mathcal{F}^{jk}\} - m\{v^j, \mathcal{F}^{ik}\}$$

$$D^i\mathcal{F}^{jk} = m\{v^j, \mathcal{F}^{ki}\} - m\{v^k, \mathcal{F}^{ji}\}$$

$$D^j\mathcal{F}^{ki} = m\{v^k, \mathcal{F}^{ij}\} - m\{v^i, \mathcal{F}^{kj}\}$$

Adding these three equations, we obtain the following result:

$$D^k\mathcal{F}^{ij} + D^i\mathcal{F}^{jk} + D^j\mathcal{F}^{ki} = 0$$

This is in fact equation (17). To conclude, then, we should stress that the Leibnitz rule for the time derivative acting on Poisson brackets has been assumed in both sections. This may not be valid in general. However, it is true if the system admits a Hamiltonian H, and the equations of motion can be written as Hamilton's equations of motion using H.

4. SPECIAL CASE OF THE $A(x, I)$ FIELD

Suppose one takes the following specific form for the $A^a(x, I)$ field:

$$A^a(x, I) = gC^{ab}(I)A_b(x)$$

It is not hard to show that the one-form given above satisfies (19). The right-hand side is

$$\delta A^{\mu b}C^{ad} - \delta_d A^{\mu a}C^{bd} = g(\delta_d C^{bk}C^{ad} - \delta_d C^{ak}C^{bd})A_k(x)$$

The left-hand side is

$$\delta_d C^{ab}A^{\mu d} = g\delta_d C^{ab}C^{dk}A_k(x)$$

Equating the coefficients of A on both sides gives the result.

Substituting the form of the derivative D^μ and this particular ansatz into (18), one obtains

$$\begin{aligned} C^{ad}\delta_d\mathcal{F}^{\mu\nu} &= (\partial^\mu - A^{\mu d}\delta_d)gC^{ab}(I)A_b^\nu(x) - (\partial^\nu - A^{\nu d}\delta_d)gC^{ab}(I)A_b^\mu(x) \\ &= gC^{ab}(\partial^\mu A_b^\nu(x) - \partial^\nu A_b^\mu(x)) - g^2(C^{dk}A_k^\mu A_b^\nu - C^{dk}A_k^\nu A_b^\mu)\delta_d C^{ab} \\ &= gC^{ab}(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu) + g^2\delta_d C^{ba}(A_k^\mu A_b^\nu - A_k^\nu A_b^\mu)C^{dk} \\ &= gC^{ab}(\partial^\mu A_b^\mu - \partial^\nu A_b^\mu) + g^2C^{kd}\delta_d C^{ba}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu) \end{aligned}$$

Substituting for $C^{kd}\delta_d C^{ba}$, we can write the second term on the right-hand side in the form

$$C^{kd}\delta_d C^{ba}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu)$$
$$= C^{ad}\delta_d C^{kb}(A_k^\mu A_b^\nu - A_b^\mu A_k^\nu) - C^{bd}\delta_d C^{ak}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu)$$

Replacing b by k and k by b, we can write this as

$$C^{kd}\delta_d C^{ba}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu)$$
$$= C^{ad}\delta_d C^{kb}(A_k^\mu A_b^\nu - A_b^\mu A_k^\nu) - C^{kd}\delta_d C^{ab}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu)$$

Solving for the common factor in this, we obtain the equation

$$C^{kd}\delta_d C^{ba}(A_b^\mu A_k^\nu - A_k^\mu A_b^\nu) = \tfrac{1}{2} C^{ad}\delta_d C^{kb}(A_k^\mu A_b^\nu - A_b^\mu A_k^\nu)$$

Substituting back into the original equation, we obtain the result

$$C^{ad}\delta_d \mathcal{F}^{\mu\nu} = gC^{ab}(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu) + \frac{g^2}{2} C^{ad}\delta_d C^{kb}(A_k^\mu A_b^\nu - A_b^\mu A_k^\nu)$$

Putting all terms on one side, we find

$$\delta_b \mathcal{F}^{\mu\nu} - g(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu) - \frac{g^2}{2} \delta_b C^{ks}(A_k^\mu A_s^\nu - A_s^\mu A_k^\nu) = 0$$

This can be written entirely in terms of forms as follows:

$$\delta_b \mathcal{F} - g dA_b(x) - \frac{g^2}{2} \delta_b C^{kl}(A_k(x) \wedge A_l(x)) = 0$$

and $\mathcal{F}$ is the two-form on Minkowski space, with components $\mathcal{F}^{\mu\nu}$. This equation can be solved by means of the equation

$$\mathcal{F}(x, I) = g dA_a I^a + \frac{g^2}{2} C^{ab}(I) A_a \wedge A_b \tag{22}$$

The dynamics of a particle in a Yang–Mills field is obtained when one sets C^{ab} equal to a sum of terms which are linear in I. Suppose one puts

$$C^{ab}(I) = c_d^{ab} I^d$$

The coefficients c_d^{ab} are the structure constants which are associated with some Lie algebra G. Then one clearly has the relations

$$c_d^{ab} = -c_d^{ba}$$
$$c_d^{bc} c_e^{ad} + c_d^{ca} c_e^{bd} + c_d^{ab} c_e^{cd} = 0$$

Moreover, A^a and $\mathcal{F}$ are also linear functions of I, $A_b = A_b(x)$ corresponding to Yang–Mills connection one-forms, and g is the coupling constant. One can write

$$\mathcal{F}(x, I) = gF_d(x)I^d$$

This is determined by equation (22) as follows:

$$\mathcal{F}(x, I) = gdA_d I^d + \frac{g^2}{2} c_d^{ab} I^d A_a \wedge A_b$$

$$= g\left(dA_d + \frac{g}{2} c_d^{ab} A_a \wedge A_b\right) I^d$$

Now $F_d = F_d(x)$ can be identified with the field strength two-form for Yang–Mills theory,

$$F_d = dA_d + \frac{g}{2} c_d^{ab} A_a \wedge A_b$$

From (17), one has for the first term on the left

$$D^\lambda \mathcal{F}^{\mu\nu} = g(\partial^\lambda F_p^{\mu\nu} + gc_p^{bd} A_b^\lambda(x) F_d^{\mu\nu}(x)) I^p$$

and with cyclic permutations, it can be seen that this generates the Bianchi identity for Yang–Mills fields, namely

$$dF_a + gc_a^{bd} A_b \wedge F_d = 0$$

It has been shown that the Lorentz force law and a pair of Maxwell equations without sources can be obtained by postulating a very simple Poisson bracket structure on the local coordinates of the phase space manifold of a particle. An elementary symmetry transformation then yields the other pair of equations. It is necessary only to postulate the particle's Poisson brackets and to assume the existence of a Hamiltonian evolution. In the case of a more general algebra, virtually everything comes from the structure of the bracket and the Jacobi identity. Other, more complicated symmetry transformations, but analogous to the one described for the Maxwell equations, may be of use in generating further equations in the generalized cases as well.

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